

SHARP HARDY-ADAMS INEQUALITIES FOR BI-LAPLACIAN ON HYPERBOLIC SPACE OF DIMENSION FOUR

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ABSTRACT. We establish sharp Hardy-Adams inequalities on hyperbolic space \mathbb{B}^4 of dimension four. Namely, we will show that for any $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that

$$\int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV = 16 \int_{\mathbb{B}^4} \frac{e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2}{(1 - |x|^2)^4} dx \leq C_\alpha.$$

for any $u \in C_0^\infty(\mathbb{B}^4)$ with

$$\int_{\mathbb{B}^4} \left(-\Delta_{\mathbb{H}} - \frac{9}{4} \right) (-\Delta_{\mathbb{H}} + \alpha) u \cdot u dV \leq 1.$$

As applications, we obtain a sharpened Adams inequality on hyperbolic space \mathbb{B}^4 and an inequality which improves the classical Adams' inequality and the Hardy inequality simultaneously. The later inequality is in the spirit of the Hardy-Trudinger-Moser inequality on a disk in dimension two given by Wang and Ye [37] and on any convex planar domain by the authors [26].

The tools of fractional Laplacian, Fourier transform and the Plancherel formula on hyperbolic and symmetric spaces play an important role in our work.

1. INTRODUCTION

Our main purpose of this article is to establish sharp Hardy-Adams inequalities on hyperbolic space in dimension four \mathbb{B}^4 .

We first recall the classical Trudinger-Moser inequality in any finite domain of Euclidean spaces. Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain and $1 \leq q \leq \frac{np}{n-kp}$. Then it is well known that the Sobolev embedding theorem tells us the embedding $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ is continuous when $kp < n$. However, in general $W_0^{1,n}(\Omega) \not\subset L^\infty(\Omega)$. Trudinger [36] established in the borderline case that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$, where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp(\beta|t|^{n/n-1}) - 1$ for some $\beta > 0$ (see also Yudovich [39], Pohozaev [34]). In 1971, Moser sharpened the Trudinger inequality in [30] by finding the optimal β :

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Theorem 1.1. *[Trudinger-Moser] Let Ω be a domain with finite measure in Euclidean n -space \mathbb{R}^n , $n \geq 2$. Then there exists a sharp constant $\beta_n = n \left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \right)^{\frac{1}{n-1}}$ such that*

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u|^{\frac{n}{n-1}}) dx \leq c_0$$

for any $\beta \leq \beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_{\Omega} |\nabla u|^n dx \leq 1$. This constant β_n is sharp in the sense that if $\beta > \beta_n$, then the above inequality can no longer hold with some c_0 independent of u .

In 1988, D. Adams extended such an inequality to high order Sobolev spaces. In fact, Adams proved the following theorem:

Theorem 1.2. *Let Ω be a domain in \mathbb{R}^n with finite n -measure and m be a positive integer less than n . There is a constant $c_0 = c_0(m, n)$ such that for all $u \in C^m(\mathbb{R}^n)$ with support contained in Ω and $\|\nabla^m u\|_{n/m} \leq 1$, the following uniform inequality holds*

$$(1.1) \quad \frac{1}{|\Omega|} \int_{\Omega} \exp(\beta_0(m, n) |u|^{n/(n-m)}) dx \leq c_0,$$

where

$$(1.2) \quad \beta_0(m, n) = \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma((m+1)/2)}{\Gamma((n-m+1)/2)} \right]^{n/(n-m)}, & m = \text{odd}; \\ \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(m/2)}{\Gamma((n-m)/2)} \right]^{n/(n-m)}, & m = \text{even}, \end{cases}$$

where ω_{n-1} is the surface measure of the unite sphere in \mathbb{R}^n .

Furthermore, the constant $\beta_0(m, n)$ in (1.1) is sharp in the sense that if $\beta_0(m, n)$ is replaced by any larger number, then the integral in (1.1) cannot be bounded uniformly by any constant.

Note that $\beta(1, n)$ coincides with Moser's value of β_n . We are particularly interested in the case $n = 4$ and $m = 2$ in this paper where $\beta_0(2, 4) = 32\pi^2$.

There have been many generalizations related to the Trudinger-Moser inequality on hyperbolic spaces and Riemannian manifolds (see e.g., [8], [16], [24], [25], [20], [21], [26], [28], [29], [38], [40]). For instance, Mancini and Sandeep [28] proved the following improved Trudinger-Moser inequalities on $\mathbb{B} = \{z = x + iy : |z| = \sqrt{x^2 + y^2} < 1\}$:

$$\sup_{u \in W_0^{1,2}(\mathbb{B}), \int_{\mathbb{B}} |\nabla u|^2 dx dy \leq 1} \int_{\mathbb{B}} \frac{e^{4\pi u^2} - 1}{(1 - |z|^2)^2} dx dy < \infty.$$

Later, Karmakar and Sandeep [16] generalize this inequality to hyperbolic space \mathbb{H}^n if n is even. In [24, 25], the first author and Tang established sharp critical and subcritical Trudinger-Moser inequalities on the high dimensional hyperbolic spaces which are different

from those in [29]. The results have been generalized by Ngô and Nguyen [31], among other results, for bi-Laplacian on hyperbolic spaces.

Wang and Ye [37] proved, among other results, an improved Trudinger-Moser inequality by combining the Hardy inequality. Their result is the following

Theorem 1.3. *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{B}} e^{\frac{4\pi u^2}{\|u\|_{\mathcal{H}}}} dx dy < C < \infty, \quad \forall u \in C_0^\infty(\mathbb{B}),$$

where $\|u\|_{\mathcal{H}} = \int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1-|z|^2)^2} dx dy$.

We note that the proof of Theorem 1.3 in [37] depends on Schwartz rearrangement argument. In the same paper, they conjecture that such Hardy-Trudinger-Moser inequality holds for bounded and convex domains with smooth boundary. Using Theorem 1.3, Mancini, Sandeep and Tintarev [29] proved, among other results, the following modified Trudinger-Moser inequality on \mathbb{B} and their proof also depends on rearrangement inequalities.

Theorem 1.4. *There exists a constant C such that for all $u \in C_0^\infty(\mathbb{B})$ with*

$$\|u\|_{\mathcal{H}} = \int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1-|z|^2)^2} dx dy \leq 1,$$

there holds

$$\int_{\mathbb{B}} \frac{(e^{4\pi u^2} - 1 - 4\pi u^2)}{(1-|x|^2)^2} dx \leq C.$$

Recently, both authors confirm in [26] that the conjecture given by Wang and Ye [37] indeed holds for any bounded and convex domain in \mathbb{R}^2 via the Riemann mapping theorem. More precisely, the authors established in [26] the following:

Theorem 1.5. *Let Ω be a bounded and convex domain in \mathbb{R}^2 . There exists a finite constant $C(\Omega) > 0$ such that*

$$\int_{\Omega} e^{\frac{4\pi u^2}{H_d(u)}} dx dy \leq C(\Omega), \quad \forall u \in C_0^\infty(\Omega),$$

where $H_d = \int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy$ and $d(z, \partial\Omega) = \min_{z_1 \in \partial\Omega} |z - z_1|$.

It then becomes a very interesting and highly nontrivial question whether Theorem 1.3 holds for higher order derivatives. In this paper we shall show this is indeed the case on n -dimensional hyperbolic spaces \mathbb{B}^n when $n = 4$.

To state our results, let us agree to some conventions. We use the Poincaré model of the hyperbolic space \mathbb{B}^n . Recall that the Poincaré model is the unit ball

$$\mathbb{B}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x| < 1\}$$

equipped with the usual Poincaré metric

$$ds^2 = \frac{4(dx_1^2 + \dots + dx_n^2)}{(1-|x|^2)^2}.$$

The hyperbolic volume element is

$$dV = \left(\frac{2}{1 - |x|^2} \right)^n dx.$$

The associated Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}} = \frac{1 - |x|^2}{4} \left\{ (1 - |x|^2) \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2(n-2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right\}.$$

The GJMS operators on \mathbb{B}^n are given by (see [10], [14])

$$P_1(P_1 + 2) \cdots (P_1 + k(k-1)), \quad k \in \mathbb{N} \setminus \{0\},$$

where $P_1 = -\Delta_{\mathbb{H}} - \frac{n(n-2)}{4}$ is the conformal Laplacian on \mathbb{B}^n . In the case $n = 4$ and $k = 2$, the GJMS operator is nothing but the Paneitz operator on \mathbb{B}^4 which satisfies (see [23])

$$P_1(P_1 + 2) = \left(\frac{1 - |x|^2}{2} \right)^4 \Delta^2,$$

where $\Delta = \sum_{i=1}^4 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on \mathbb{R}^4 . Therefore, for $u \in C_0^\infty(\mathbb{B}^4)$,

$$(1.3) \quad \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}})(-\Delta_{\mathbb{H}} + 2)u \cdot u dV = \int_{\mathbb{B}^4} |\Delta u|^2 dx.$$

It is known that the spectral gap of $-\Delta_{\mathbb{H}}$ on $L^2(\mathbb{B}^n)$ is $\frac{(n-1)^2}{4}$ (see e.g. [27]), i.e.

$$(1.4) \quad \int_{\mathbb{B}^n} |\nabla_{\mathbb{H}} u|^2 dV \geq \frac{(n-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV, \quad u \in C_0^\infty(\mathbb{B}^n),$$

and the constant $\frac{(n-1)^2}{4}$ is sharp. Therefore, by (1.3), we have in dimension four,

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx = \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}})(-\Delta_{\mathbb{H}} + 2)u \cdot u dV \geq \frac{9}{16} \int_{\mathbb{B}^4} u^2 dV = 9 \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx.$$

Furthermore, the constant 9 in above inequality is also sharp (see e.g. [33]).

One of the main results of this paper is the following

Theorem 1.6. *Let $\alpha > 0$. Then there exists a constant $C_\alpha > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^4)$ with*

$$\int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4)(-\Delta_{\mathbb{H}} + \alpha)u \cdot u dV \leq 1,$$

there holds

$$\int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV = 16 \int_{\mathbb{B}^4} \frac{e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2}{(1 - |x|^2)^4} dx \leq C_\alpha.$$

Choosing $\alpha = \frac{1}{4}$ and combining (1.3) and Theorem 1.6, we have the following Hardy-Adams inequalities

Theorem 1.7. *There exists a constant $C_1 > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^4)$ with*

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx - 9 \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \leq 1,$$

there holds

$$\int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1 - 32u^2) dV = 16 \int_{\mathbb{B}^4} \frac{e^{32\pi^2 u^2} - 1 - 32u^2}{(1 - |x|^2)^4} dx \leq C_1.$$

Theorem 1.7 implies the following improved Adams inequalities.

Theorem 1.8. *Let $\lambda < 9$. Then there exists a constant $C_\lambda > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^4)$ with*

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx - \lambda \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \leq 1,$$

there holds

$$\int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1) dV = 16 \int_{\mathbb{B}^4} \frac{e^{32\pi^2 u^2} - 1}{(1 - |x|^2)^4} dx \leq C_\lambda.$$

As an application of the above theorem, we also have the following Hardy-Adams inequality which is a higher dimensional analogue of the Hardy-Trudinger-Moser inequality given by Wang and Ye [37] and the authors [26].

Theorem 1.9. *There exists a constant $C_3 > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^4)$ with*

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx - 9 \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \leq 1,$$

there holds

$$\int_{\mathbb{B}^4} e^{32\pi^2 u^2} dx \leq C_3.$$

Obviously, this theorem is stronger than the classical Adams inequality in [1] which is stated as:

$$\int_{\mathbb{B}^4} e^{32\pi^2 u^2} dx \leq C_3$$

under the more restrictive constraint $\int_{\mathbb{B}^4} |\Delta u|^2 dx \leq 1$ for all $u \in C_0^\infty(\mathbb{B}^4)$.

We remark that in a forthcoming paper, we will establish the Hardy-Adams inequalities on hyperbolic spaces of all dimensions n when n is even and $n \geq 4$.

The organization of the paper is as follows: In Section 2, we review some necessary preliminaries on the hyperbolic spaces of Poincaré model on the unit ball \mathbb{B}^n , the convolution, fractional Laplacian and Fourier transform on the hyperbolic space \mathbb{B}^n defined using the the Harish-Chandra \mathfrak{c} -function; Section 3 gives the pointwise estimates of Green function for fractional Laplacians; Section 4 establishes the symmetrization functions of the Green functions; Section 5 offers the proofs of our main results, namely Theorems 1.6, 1.7, 1.8 and 1.9.

2. PRELIMINARIES ON FOURIER TRANSFORM AND FRACTIONAL LAPLACIANS ON HYPERBOLIC SPACES

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [2, 9, 12, 13, 15, 22] for more information about this subject.

Recall that the Poincaré model is the unit ball

$$\mathbb{B}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x| < 1\}$$

equipped with the usual Poincaré metric

$$ds^2 = \frac{4(dx_1^2 + \dots + dx_n^2)}{(1 - |x|^2)^2}.$$

The distance from origin to $x \in \mathbb{B}^n$ is

$$\rho(x) = \log \frac{1 + |x|}{1 - |x|}.$$

and the polar coordinate is

$$\int_{\mathbb{B}^n} f dV = \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} f \cdot (\sinh \rho)^{n-1} d\rho d\sigma, \quad f \in L^1(\mathbb{B}^n).$$

For each $a \in \mathbb{B}^n$, we define the Möbius transformations T_a by (see e.g. [2, 15])

$$T_a(x) = \frac{|x - a|^2 a - (1 - |a|^2)(x - a)}{1 - 2x \cdot a + |x|^2 |a|^2},$$

where $x \cdot a = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$ denotes the scalar product in \mathbb{R}^n . Using the Möbius transformations, the associated distance from x to y in \mathbb{B}^n is

$$\rho(x, y) = \rho(T_x(y)) = \rho(T_y(x)).$$

Also using the Möbius transformations, we can define the convolution of measurable functions f and g on \mathbb{B}^n by (see e.g. [22])

$$(2.1) \quad (f * g)(x) = \int_{\mathbb{B}^n} f(y) g(T_x(y)) dV_y$$

provided this integral exists. It is easy to check that

$$f * g = g * f.$$

Furthermore, if g is radial, i.e. $g = g(\rho)$, then (see e.g. [22])

$$(2.2) \quad (f * g) * h = f * (g * h)$$

provided $f, g, h \in L^1(\mathbb{B}^n)$

Denote by $e^{t\Delta_{\mathbb{H}}}$ the heat kernel on \mathbb{B}^n . It is well known that $e^{t\Delta_{\mathbb{H}}}$ depends only on t and $\rho(x, y)$. In fact, $e^{t\Delta_{\mathbb{H}}}$ is given explicitly by the following formulas (see e.g. [7, 11]):

- If $n = 2m$, then

$$e^{t\Delta_{\mathbb{H}}} = (2\pi)^{-\frac{n+1}{2}} t^{-\frac{1}{2}} e^{-\frac{(n-1)^2}{4}t} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^m e^{-\frac{r^2}{4t}} dr;$$

- If $n = 2m + 1$, then

$$e^{t\Delta_{\mathbb{H}}} = 2^{-m-1} \pi^{-m-1/2} t^{-\frac{1}{2}} e^{-\frac{(n-1)^2}{4}t} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^m e^{-\frac{\rho^2}{4t}}.$$

Finally, we review some basic facts about Fourier transform and fractional Laplacian on hyperbolic space. Set

$$e_{\lambda, \zeta}(x) = \left(\frac{\sqrt{1 - |x|^2}}{|x - \zeta|} \right)^{n-1+i\lambda}, \quad x \in \mathbb{B}^n, \quad \lambda \in \mathbb{R}, \quad \zeta \in \mathbb{S}^{n-1}.$$

The Fourier transform of a function f on \mathbb{B}^n can be defined as

$$\widehat{f}(\lambda, \zeta) = \int_{\mathbb{B}^n} f(x) e_{-\lambda, \zeta}(x) dV$$

provided this integral exists. If $g \in C_0^\infty(\mathbb{B}^n)$ is radial, then

$$\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}.$$

Moreover, the following inversion formula holds for $f \in C_0^\infty(\mathbb{B}^n)$ (see [22]):

$$f(x) = D_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \zeta) e_{\lambda, \zeta}(x) |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\zeta),$$

where $D_n = \frac{1}{2^{3-n}\pi|\mathbb{S}^{n-1}|}$ and $\mathfrak{c}(\lambda)$ is the Harish-Chandra \mathfrak{c} -function given by (see [22])

$$\mathfrak{c}(\lambda) = \frac{2^{n-1-i\lambda} \Gamma(n/2) \Gamma(i\lambda)}{\Gamma(\frac{n-1+i\lambda}{2}) \Gamma(\frac{1+i\lambda}{2})}.$$

Similarly, there holds the Plancherel formula:

$$\int_{\mathbb{B}^n} |f(x)|^2 dV = D_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\zeta).$$

Since $e_{\lambda, \zeta}(x)$ is an eigenfunction of $\Delta_{\mathbb{H}}$ with eigenvalue $-\frac{(n-1)^2 + \lambda^2}{4}$, it is easy to check that, for $f \in C_0^\infty(\mathbb{B}^n)$,

$$\widehat{\Delta_{\mathbb{H}} f}(\lambda, \zeta) = -\frac{(n-1)^2 + \lambda^2}{4} \widehat{f}(\lambda, \zeta).$$

Therefore, in analogy with the Euclidean setting, we define the fractional Laplacian on hyperbolic space as follows:

$$(-\widehat{\Delta_{\mathbb{H}}})^\gamma f(\lambda, \zeta) = \left(\frac{(n-1)^2 + \lambda^2}{4} \right)^\gamma \widehat{f}(\lambda, \zeta), \quad \gamma \in \mathbb{R}.$$

For more information about fractional Laplacian on hyperbolic space and symmetric spaces, we refer to [3, 5].

3. SHARP ESTIMATES FOR THE GREEN FUNCTION AND FRACTIONAL POWER

In the rest of paper, we shall fix $n = 4$. In what follows, $a \lesssim b$ will stand for $a \leq Cb$ with some positive absolute constant C .

Since the work of Adams [1], it has been a standard approach to establish sharp Trudinger-Moser and Adams inequalities in different settings including both Riemannian and sub-Riemannian settings such as on the Heisenberg groups by using the sharp pointwise estimates of Green's functions together with using O'Neil's lemma of convolutions [32]. We shall not go to the details here. To this end, we will derive the pointwise estimates for Green's functions of powers of Laplacians in the hyperbolic spaces to establish our Hardy-Adams inequalities.

Recall that the heat kernel

$$e^{t\Delta_{\mathbb{H}}} = (2\pi)^{-\frac{5}{2}} t^{-\frac{1}{2}} e^{-\frac{9}{4}t} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^2 e^{-\frac{r^2}{4t}} dr$$

and the Mellin type expression on hyperbolic space (see e.g. [4], Section 4.2)

$$(-\Delta_{\mathbb{H}} + \alpha)^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^{+\infty} t^{\sigma-1} e^{t(\Delta_{\mathbb{H}} - \alpha)} dt, \quad \alpha \geq -9/4, \quad 3/2 > \sigma > 0.$$

We have

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-1} &= \frac{1}{(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-\frac{1}{2}} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^2 e^{-\frac{r^2}{4t}} dt \\ &= \frac{1}{(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} dr \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right) \int_0^{+\infty} \frac{r}{2 \sinh r} t^{-\frac{3}{2}} e^{-\frac{r^2}{4t}} dt \\ &= \frac{1}{(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right) \frac{\Gamma(\frac{1}{2})}{\sinh r} dr \\ &= \frac{1}{4\sqrt{2}\pi^2} \int_{\rho}^{+\infty} \frac{\cosh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \end{aligned}$$

Here we use the fact $\Gamma(1/2) = \sqrt{\pi}$.

Lemma 3.1. *There holds, for $\rho > 0$,*

$$(-\Delta_{\mathbb{H}} - 9/4)^{-1} \leq \frac{1}{4\pi^2 \cosh \frac{\rho}{2} \sinh^2 \rho} + \frac{1}{4\pi^2 \cosh \frac{\rho}{2} \sinh \rho}.$$

Proof. Using the substitution $t = \sqrt{\cosh r - \cosh \rho}$, we have

$$\begin{aligned}
(-\Delta_{\mathbb{H}} - 9/4)^{-1} &= \frac{1}{4\sqrt{2}\pi^2} \int_{\rho}^{+\infty} \frac{\cosh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \\
&= \frac{1}{2\sqrt{2}\pi^2} \int_0^{+\infty} \frac{t^2 + \cosh \rho}{[(t^2 + \cosh \rho)^2 - 1]^{\frac{3}{2}}} dt \\
&= \frac{1}{2\sqrt{2}\pi^2} \int_0^{+\infty} \frac{t^2 + \cosh \rho - 1}{[(t^2 + \cosh \rho)^2 - 1]^{\frac{3}{2}}} dt + \frac{1}{2\sqrt{2}\pi^2} \int_0^{+\infty} \frac{1}{[(t^2 + \cosh \rho)^2 - 1]^{\frac{3}{2}}} dt \\
&=: (I) + (II)
\end{aligned}$$

where

$$\begin{aligned}
(I) &= \frac{1}{2\sqrt{2}\pi^2} \int_0^{+\infty} \frac{t^2 + \cosh \rho - 1}{[(t^2 + \cosh \rho)^2 - 1]^{\frac{3}{2}}} dt \\
&= \frac{1}{2\sqrt{2}\pi^2} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)^{\frac{3}{2}} (t^2 + \cosh \rho - 1)^{\frac{1}{2}}} dt \\
&\leq \frac{1}{2\sqrt{2}\pi^2} \frac{1}{\sqrt{\cosh \rho - 1}} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)^{\frac{3}{2}}} dt \\
&= \frac{1}{2\sqrt{2}\pi^2} \frac{1}{\sqrt{\cosh \rho - 1}} \cdot \frac{1}{\cosh \rho + 1} \frac{t}{\sqrt{t^2 + \cosh \rho + 1}} \Big|_0^{\infty} \\
&= \frac{1}{2\sqrt{2}\pi^2} \frac{1}{\sinh \rho \sqrt{\cosh \rho + 1}} = \frac{1}{4\pi^2 \cosh \frac{\rho}{2} \sinh \rho}; \\
(II) &= \frac{1}{2\sqrt{2}\pi^2} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)^{\frac{3}{2}} (t^2 + \cosh \rho - 1)^{\frac{3}{2}}} dt \\
&\leq \frac{1}{2\sqrt{2}\pi^2} \frac{1}{(\cosh \rho + 1)^{\frac{3}{2}}} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho - 1)^{\frac{3}{2}}} dt \\
&= \frac{1}{2\sqrt{2}\pi^2} \frac{1}{(\cosh \rho + 1)^{\frac{3}{2}}} \cdot \frac{1}{\cosh \rho - 1} \frac{t}{\sqrt{t^2 + \cosh \rho - 1}} \Big|_0^{\infty} \\
&= \frac{1}{2\sqrt{2}\pi^2} \frac{1}{\sinh^2 \rho \sqrt{\cosh \rho + 1}} = \frac{1}{4\pi^2 \cosh \frac{\rho}{2} \sinh^2 \rho}.
\end{aligned}$$

The desired result follows. \square

Also via the heat kernel and the Mellin type expression, the fractional power

$$\begin{aligned}
(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} &= \frac{1}{\Gamma(1/2)} \int_0^{+\infty} t^{-\frac{1}{2}} e^{t(\Delta_{\mathbb{H}} - \alpha)} dt \\
(3.1) \quad &= \frac{1}{\sqrt{\pi}(\sqrt{2}\pi)^5} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-1} e^{-\alpha t - \frac{9}{4}t} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^2 e^{-\frac{r^2}{4t}} dt,
\end{aligned}$$

where $\alpha \geq -9/4$. It is easy to check that if $\alpha > -9/4$, then

$$(3.2) \quad (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} \leq (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}.$$

Furthermore, we have the following estimates of $(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}$.

Lemma 3.2. *There holds, for $\rho > 0$*

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{1}{16\pi^2(1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} + \frac{\sqrt{2}}{4\pi^2\sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho}; \\ (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{8}{\sqrt{\pi}(\sqrt{2\pi})^5} \cdot \frac{1}{\rho\sqrt{\cosh \rho + 1}(\cosh \rho - 1)}. \end{aligned}$$

Proof. By (3.1),

$$(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-1} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^2 e^{-\frac{r^2}{4t}} dt.$$

Notice that, for $r > 0$,

$$\begin{aligned} \int_0^{+\infty} t^{-1} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^2 e^{-\frac{r^2}{4t}} dt &= \int_0^{+\infty} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right) \left(\frac{r}{\sinh r} \cdot \frac{1}{2t^2} e^{-\frac{r^2}{4t}} \right) dt \\ &= \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right) \left(\frac{r}{2\sinh r} \cdot \int_0^{+\infty} t^{-2} e^{-\frac{r^2}{4t}} dt \right) \\ &= \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right) \frac{2}{r \sinh r} = \frac{2}{r^2 \sinh^2 r} + 2 \frac{\cosh r}{r \sinh^3 r}, \end{aligned}$$

we have,

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &= \frac{2}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{1}{\sqrt{\cosh r - \cosh \rho}} \left(\frac{1}{r^2 \sinh r} + \frac{\cosh r}{r \sinh^2 r} \right) dr \\ (3.3) \quad &\leq \frac{2}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{1}{\sqrt{\cosh r - \cosh \rho}} \left(\frac{\cosh r}{r \sinh^2 r} + \frac{\cosh r}{r \sinh^2 r} \right) dr \\ &= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\cosh r}{r \sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr. \end{aligned}$$

To get the first inequality in (3.3), we use the inequality $\frac{1}{r} \leq \coth r (r > 0)$. Therefore,

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\cosh r}{r \sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \\ &\leq \frac{2}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\cosh r (\cosh r + 1)}{\sinh^3 r \sqrt{\cosh r - \cosh \rho}} dr. \end{aligned}$$

Here we use the fact $\frac{1}{r} \leq \frac{1}{2} \coth \frac{r}{2} = \frac{1+\cosh r}{2\sinh r}$ ($r > 0$). Using the substitution $t = \sqrt{\cosh r - \cosh \rho}$ yields

$$\begin{aligned}
(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{2}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\cosh r(\cosh r + 1)}{\sinh^3 r \sqrt{\cosh r - \cosh \rho}} dr \\
&= \frac{2}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\cosh r}{\sinh r(\cosh r - 1)\sqrt{\cosh r - \cosh \rho}} dr \\
&= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_0^{+\infty} \frac{t^2 + \cosh \rho}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)^2} dt \\
&= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_0^{+\infty} \frac{t^2 + \cosh \rho - 1}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)^2} dt \\
&\quad + \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)^2} dt \\
&=: (III) + (IV),
\end{aligned}$$

where

$$\begin{aligned}
(III) &= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_0^{+\infty} \frac{t^2 + \cosh \rho - 1}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)^2} dt \\
&= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)} dt \\
&\leq \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \frac{1}{1 + \cosh \rho} \cdot \int_0^{+\infty} \frac{1}{t^2 + \cosh \rho - 1} dt \\
&= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \frac{1}{1 + \cosh \rho} \cdot \frac{1}{\sqrt{\cosh \rho - 1}} \frac{\pi}{2} \\
&= \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho}; \\
(IV) &= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)^2} dt \\
&\leq \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \frac{1}{(1 + \cosh \rho)} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho - 1)^2} dt \\
&= \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \frac{1}{(1 + \cosh \rho)} \frac{1}{(\cosh \rho - 1)^{3/2}} \frac{\pi}{4} \\
&= \frac{1}{16\pi^2(1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}}.
\end{aligned}$$

Thus,

$$(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} = (III) + (IV) \leq \frac{1}{16\pi^2(1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} + \frac{\sqrt{2}}{4\pi^2\sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho}.$$

On the other hand, for $\rho > 0$, we have, by (3.3),

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\cosh r}{r \sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \\ &\leq \frac{4}{\sqrt{\pi}(\sqrt{2\pi})^5 \rho} \int_{\rho}^{+\infty} \frac{\cosh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr. \end{aligned}$$

Also using the substitution $t = \sqrt{\cosh r - \cosh \rho}$, we have

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{8}{\sqrt{\pi}(\sqrt{2\pi})^5 \rho} \int_0^{+\infty} \frac{t^2 + \cosh \rho}{[(t^2 + \cosh \rho)^2 - 1]^{\frac{3}{2}}} dt \\ &\leq \frac{8}{\sqrt{\pi}(\sqrt{2\pi})^5 \rho} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho - 1)^{\frac{1}{2}} (t^2 + \cosh \rho - 1)^{\frac{3}{2}}} dt \\ &\leq \frac{8}{\sqrt{\pi}(\sqrt{2\pi})^5 \rho} \cdot \frac{1}{\sqrt{\cosh \rho + 1}} \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho - 1)^{\frac{3}{2}}} dt \\ &= \frac{8}{\sqrt{\pi}(\sqrt{2\pi})^5} \cdot \frac{1}{\rho \sqrt{\cosh \rho + 1} (\cosh \rho - 1)}. \end{aligned}$$

The proof of Lemma 3.2 is then completed. \square

Corollary 3.3. There holds, for $\rho > 0$,

$$(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} \leq \frac{1}{4\pi^2 \sinh^3 \rho} + \left(\frac{1}{8\pi^2 \cosh \rho \cosh \rho/2} + \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \right) \frac{1}{\sinh \rho}$$

and

$$(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} \lesssim \rho^{-1} e^{-\frac{3}{2}\rho}, \quad \rho > 1.$$

Proof. By Lemma 3.2,

$$\begin{aligned}
(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{1}{16\pi^2(1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} \cdot \left(\frac{1}{\cosh \rho} + \frac{\cosh \rho - 1}{\cosh \rho} \right) + \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho} \\
&= \frac{1}{16\pi^2 \cosh \rho (1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} + \frac{\cosh \rho - 1}{16\pi^2 \cosh \rho (1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} + \\
&\quad \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho} \\
&= \frac{1}{16\pi^2 \cosh \rho (1 + \cosh \rho)} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} + \frac{1}{8\pi^2 \cosh \rho (1 + \cosh \rho)} \cdot \frac{1}{\sinh \frac{\rho}{2}} + \\
&\quad \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho}.
\end{aligned}$$

Since $1 + \cosh \rho = 2 \cosh^2 \rho/2$ and $\cosh \rho/2 \leq \cosh \rho$, we have

$$\begin{aligned}
(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{1}{16\pi^2 \cosh \rho/2 \cdot 2 \cosh^2 \rho/2} \cdot \frac{1}{\sinh^3 \frac{\rho}{2}} + \frac{1}{8\pi^2 \cosh \rho \cdot 2 \cosh^2 \rho/2} \cdot \frac{1}{\sinh \frac{\rho}{2}} + \\
&\quad \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \cdot \frac{1}{\sinh \rho} \\
&= \frac{1}{4\pi^2 \sinh^3 \rho} + \left(\frac{1}{8\pi^2 \cosh \rho \cosh \rho/2} + \frac{\sqrt{2}}{4\pi^2 \sqrt{1 + \cosh \rho}} \right) \frac{1}{\sinh \rho}.
\end{aligned}$$

Also by Lemma 3.2,

$$\begin{aligned}
(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} &\leq \frac{8}{\sqrt{\pi}(\sqrt{2\pi})^5} \cdot \frac{1}{\rho \sqrt{\cosh \rho + 1}(\cosh \rho - 1)} \\
&\lesssim \frac{1}{\rho e^{\frac{3}{2}\rho}}, \quad \rho > 1.
\end{aligned}$$

This completes the proof of Corollary 3.3. □

Lemma 3.4. *Let $\alpha > 0$. Then there exist $\alpha_0 > 0$ such that*

$$(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} \lesssim e^{-(3+\alpha_0)\rho}, \quad \rho > 1.$$

Proof. We have

$$\begin{aligned}
(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} &= \frac{1}{\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-1} e^{-\alpha t - \frac{9}{4}t} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^2 e^{-\frac{r^2}{4t}} dt \\
&= \frac{1}{2\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{r \cosh r - \sinh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-2} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt + \\
&\quad \frac{1}{4\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{r^2}{\sinh r \sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-3} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt \\
&=: (V) + (VI),
\end{aligned}$$

where

$$\begin{aligned}
(V) &= \frac{1}{2\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{r \cosh r - \sinh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-2} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt \\
&\lesssim \int_{\rho}^{+\infty} \frac{r \cosh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-2} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt; \\
(VI) &= \frac{1}{4\sqrt{\pi}(\sqrt{2\pi})^5} \int_{\rho}^{+\infty} \frac{r^2}{\sinh r \sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-3} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt.
\end{aligned}$$

Notice that $\forall \epsilon \in (0, 1)$, $e^{-\alpha t - \frac{9}{4}t} e^{-\frac{(1-\epsilon)r^2}{4t}}$ has as a maximum value $e^{-\sqrt{(\alpha + \frac{9}{4})(1-\epsilon)}r}$. We have,

$$\begin{aligned}
\int_0^{+\infty} t^{-2} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt &\leq e^{-\sqrt{(\alpha + \frac{9}{4})(1-\epsilon)}r} \int_0^{+\infty} t^{-2} e^{-\frac{\epsilon r^2}{4t}} dt = e^{-\sqrt{(\alpha + \frac{9}{4})(1-\epsilon)}r} \cdot \frac{4}{\epsilon r^2}; \\
\int_0^{+\infty} t^{-3} e^{-\alpha t - \frac{9}{4}t} e^{-\frac{r^2}{4t}} dt &\leq e^{-\sqrt{(\alpha + \frac{9}{4})(1-\epsilon)}r} \int_0^{+\infty} t^{-3} e^{-\frac{\epsilon r^2}{4t}} dt = e^{-\sqrt{(\alpha + \frac{9}{4})(1-\epsilon)}r} \cdot \frac{16}{\epsilon r^4}.
\end{aligned}$$

Choose $\epsilon_0 \in (0, 1)$ such that $\alpha_0 = \sqrt{(\alpha + \frac{9}{4})(1 - \epsilon_0)} - \frac{3}{2} > 0$. Then, for $\rho > 1$,

$$\begin{aligned}
(V) &\lesssim \int_{\rho}^{+\infty} \frac{r \cosh r}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} \cdot \frac{1}{r^2 e^{\frac{3}{2}r + \alpha_0 r}} dr \\
&\lesssim \int_{\rho}^{+\infty} \frac{1}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} \cdot \frac{1}{r e^{\frac{1}{2}r + \alpha_0 r}} dr \\
&\lesssim \frac{1}{\rho e^{\frac{1}{2}\rho + \alpha_0 \rho}} \int_{\rho}^{+\infty} \frac{1}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \\
&\leq \frac{1}{e^{\frac{1}{2}\rho + \alpha_0 \rho}} \int_{\rho}^{+\infty} \frac{1}{\sinh^2 r \sqrt{\cosh r - \cosh \rho}} dr \\
&\leq \frac{1}{e^{\frac{1}{2}\rho + \alpha_0 \rho} \sinh \rho} \int_{\rho}^{+\infty} \frac{1}{\sinh r \sqrt{\cosh r - \cosh \rho}} dr;
\end{aligned}$$

$$\begin{aligned}
(VI) &\lesssim \int_{\rho}^{+\infty} \frac{1}{\sinh r \sqrt{\cosh r - \cosh \rho}} \cdot \frac{1}{r^2 e^{\frac{3}{2}r + \alpha_0 r}} dr \\
&\lesssim \frac{1}{e^{\frac{3}{2}\rho + \alpha_0 \rho}} \int_{\rho}^{+\infty} \frac{1}{\sinh r \sqrt{\cosh r - \cosh \rho}} dr.
\end{aligned}$$

On the other hand, using the substitution $t = \sqrt{\cosh r - \cosh \rho}$, we have

$$\begin{aligned}
\int_{\rho}^{+\infty} \frac{1}{\sinh r \sqrt{\cosh r - \cosh \rho}} dr &= 2 \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho)^2 - 1} dt \\
&= 2 \int_0^{+\infty} \frac{1}{(t^2 + \cosh \rho + 1)(t^2 + \cosh \rho - 1)} dt \\
&\leq \frac{2}{\cosh \rho + 1} \int_0^{+\infty} \frac{1}{t^2 + \cosh \rho - 1} dt \\
&= \frac{2}{\cosh \rho + 1} \cdot \frac{1}{\sqrt{\cosh \rho - 1}} \frac{\pi}{2}.
\end{aligned}$$

Therefore, for $\rho > 1$,

$$\begin{aligned}
(V) &\lesssim \frac{1}{e^{\frac{1}{2}\rho + \alpha_0 \rho} \sinh \rho} \cdot \frac{2}{\cosh \rho + 1} \cdot \frac{1}{\sqrt{\cosh \rho - 1}} \frac{\pi}{2} \lesssim \frac{1}{e^{(3+\alpha_0)\rho}}; \\
(VI) &\lesssim \frac{1}{e^{\frac{3}{2}\rho + \alpha_0 \rho}} \cdot \frac{2}{\cosh \rho + 1} \cdot \frac{1}{\sqrt{\cosh \rho - 1}} \frac{\pi}{2} \lesssim \frac{1}{e^{(3+\alpha_0)\rho}}.
\end{aligned}$$

Thus

$$(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} = (V) + (VI) \lesssim \frac{1}{e^{(3+\alpha_0)\rho}}, \quad \rho > 1.$$

The proof is then completed. \square

4. REARRANGEMENT

We now recall the rearrangement of a real functions on \mathbb{B}^4 . Suppose f is a real function on \mathbb{B}^4 . The non-increasing rearrangement of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\},$$

where

$$\lambda_f(s) = |\{x \in \mathbb{B}^4 : |f(x)| > s\}| = \int_{\{x \in \mathbb{B}^4 : |f(x)| > s\}} \left(\frac{2}{1 - |x|^2} \right)^4 dx.$$

Here we use the notation $|\Sigma|$ for the measure of a measurable set $\Sigma \subset \mathbb{B}^4$.

Lemma 4.1. *There exists a constant $A_1 > 0$ such that*

$$[(-9/4 - \Delta_{\mathbb{H}})^{-1}]^*(t) \leq \frac{1}{4\sqrt{2}\pi\sqrt{t}}(1 + A_1 t^{\frac{1}{4}}), \quad t > 0.$$

Proof. Set $\phi(\rho) = (-9/4 - \Delta_{\mathbb{H}})^{-1}$. Define, for any $s > 0$,

$$(4.1) \quad \lambda_{\phi}(s) = \int_{\{\phi(\rho) > s\}} dV = |\mathbb{S}^3| \int_0^{\rho_s} \sinh^3 \rho d\rho = 2\pi^2 \int_0^{\rho_s} \sinh^3 \rho d\rho,$$

where ρ_s is the solution of equation

$$(4.2) \quad \phi(\rho) = s.$$

Therefore, since $\phi^*(t) = \inf\{s > 0 : \lambda_{\phi}(s) \leq t\}$, we have

$$(4.3) \quad t = \lambda_{\phi}(\phi^*(t)) = 2\pi^2 \int_0^{\rho_{\phi^*(t)}} \sinh^3 \rho d\rho,$$

where $\rho_{\phi^*(t)}$ satisfies

$$(4.4) \quad \phi(\rho_{\phi^*(t)}) = \phi^*(t).$$

Notice that

$$(4.5) \quad t = 2\pi^2 \int_0^{\rho_{\phi^*(t)}} \sinh^3 \rho d\rho \leq 2\pi^2 \int_0^{\rho_{\phi^*(t)}} \sinh^3 \rho \cosh \rho d\rho = \frac{\pi^2}{2} \sinh^4 \rho_{\phi^*(t)};$$

$$(4.6) \quad t = 2\pi^2 \int_0^{\rho_{\phi^*(t)}} \sinh^3 \rho d\rho \leq 2\pi^2 \int_0^{\rho_{\phi^*(t)}} e^{3\rho} d\rho = \frac{2\pi^2}{3} e^{3\rho_{\phi^*(t)}}.$$

We have, by (4.4)-(4.5) and Lemma 3.1,

$$\begin{aligned} \phi^*(t) &\leq \frac{1}{4\sqrt{2}\pi\sqrt{t} \cdot \cosh \frac{\rho_{\phi^*(t)}}{2}} + \frac{1}{4\pi^2 \cosh \frac{\rho_{\phi^*(t)}}{2}} \left(\frac{\pi^2}{2t} \right)^{\frac{1}{4}} \\ &\leq \frac{1}{4\sqrt{2}\pi\sqrt{t}} (1 + A_1 t^{\frac{1}{4}}), \end{aligned}$$

where $A_1 = 2^{\frac{1}{4}}\pi^{-\frac{1}{2}}$. The desired result follows. \square

Similarly, by Corollary 3.3 and Lemma 3.4, we have the following (we omit the proof because it is completely the same to that of Lemma 4.1).

Lemma 4.2. *Let $\alpha > 0$ and ϵ_0 be in Lemma 3.3. Then there exists a constant $A_2 > 0$ such that*

$$\begin{aligned} [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(t) &\leq \frac{2^{\frac{1}{4}}}{8\sqrt{\pi}t^{\frac{3}{4}}}(1 + A_2\sqrt{t}), \quad t > 0; \\ [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(t) &\lesssim \frac{1}{\sqrt{t} \ln t}, \quad t > 2; \\ [(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(t) &\lesssim \frac{1}{t^{1+\frac{1}{3}\epsilon_0}}, \quad t > 2. \end{aligned}$$

Since for $\alpha > -9/4$,

$$(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} \leq (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}},$$

we have, by Lemma 4.2,

$$(4.7) \quad [(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(t) \leq [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(t) \leq \frac{2^{\frac{1}{4}}}{8\sqrt{\pi}t^{\frac{3}{4}}}(1 + A_2\sqrt{t}), \quad t > 0.$$

Lemma 4.3. *Let $\alpha > 0$ and ϵ_0 be in Lemma 3.3. Then*

$$(4.8) \quad [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(t) \leq \frac{1}{4\sqrt{2}\pi\sqrt{t}}(1 + A_1t^{\frac{1}{4}}), \quad t > 0$$

and

$$(4.9) \quad [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(t) \lesssim \frac{1}{\sqrt{t} \ln t}, \quad t > 2.$$

Proof. Since $(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} = (-\Delta_{\mathbb{H}} - 9/4)^{-1}$, we have, by (3.2),

$$\begin{aligned} (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} &\leq (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} \\ &= (-\Delta_{\mathbb{H}} - 9/4)^{-1}. \end{aligned}$$

Therefore, by Lemma 4.1, we get (4.8).

Now we prove (4.9). By O'Neil's lemma (see [32]),

$$\begin{aligned} &[(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(t) \\ (4.10) \quad &\leq \frac{1}{t} \int_0^t [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(s) ds \cdot \int_0^t [(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(s) ds + \\ &\int_t^\infty [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(s) [(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(s) ds. \end{aligned}$$

By Lemma 4.2, it is easy to check

$$(4.11) \quad \int_0^t [(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(s) ds \lesssim \int_0^2 \frac{2^{\frac{1}{4}}}{8\sqrt{\pi}s^{\frac{3}{4}}}(1 + A_2\sqrt{s}) ds + \int_2^t s^{-1-\frac{1}{3}\epsilon_0} ds \lesssim 1, \quad t > 2.$$

Also by Lemma 4.2 and (4.7), we have, for $t > 2$,

$$\begin{aligned} (4.12) \quad &\int_0^t [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(s) ds \lesssim \int_0^2 \frac{2^{\frac{1}{4}}}{8\sqrt{\pi}s^{\frac{3}{4}}}(1 + A_2\sqrt{s}) ds + \int_2^t \frac{1}{\sqrt{s} \ln s} ds \\ &\lesssim 1 + \int_2^t \frac{1}{\sqrt{s} \ln s} ds. \end{aligned}$$

Notice that, by L'Hospital's law,

$$\lim_{t \rightarrow \infty} \frac{\int_2^t \frac{1}{\sqrt{s} \ln s} ds}{\frac{\sqrt{t}}{\ln t}} = \lim_{t \rightarrow \infty} \frac{2}{1 - \frac{1}{\ln t}} = 2.$$

We have, $\int_2^t \frac{1}{\sqrt{s} \ln s} ds \lesssim \frac{\sqrt{t}}{\ln t}$, $t > 2$. Therefore, by (4.12),

$$(4.13) \quad \int_0^t [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(s) ds \lesssim 1 + \frac{\sqrt{t}}{\ln t} \lesssim \frac{\sqrt{t}}{\ln t}, \quad t > 2.$$

Similarly, by Lemma 4.2, for $t > 2$,

$$(4.14) \quad \int_t^\infty [(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}]^*(s) [(-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(s) ds \lesssim \int_t^\infty \frac{1}{s^{\frac{3}{2} + \frac{1}{3}\epsilon_0} \ln s} ds \lesssim \frac{1}{t^{\frac{1}{2} + \frac{1}{3}\epsilon_0} \ln t}.$$

Combing (4.10), (4.11), (4.13) and (4.14) yields, for $t > 2$,

$$[(-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}]^*(t) \lesssim \frac{1}{t} \cdot \frac{\sqrt{t}}{\ln t} \cdot 1 + \frac{1}{t^{\frac{1}{2} + \frac{1}{3}\epsilon_0} \ln t} \lesssim \frac{1}{\sqrt{t} \ln t}.$$

The desired result follows. \square

5. PROOFS OF MAIN THEOREMS

Firstly, we shall prove Theorem 1.4. The main idea is to decompose the whole space by the level set of the functions under consideration and derive the global inequality on the whole space from the local ones. This idea was initially developed different settings by Lam and the first author to derive a global Trudinger-Moser inequality from a local one (see [17, 18]).

Proof of Theorem 1.6 Let $u \in C_0^\infty(\mathbb{B})$ be such that

$$\int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + \alpha) u \cdot u dV \leq 1,$$

We have, by (1.4),

$$(5.1) \quad \begin{aligned} & \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + \alpha) u \cdot u dV \\ & \geq \left(\frac{9}{4} + \alpha \right) \left(\int_{\mathbb{B}^4} |\nabla_{\mathbb{H}} u|^2 dV - \frac{9}{4} \int_{\mathbb{B}^4} |u|^2 dV \right) \\ & \geq \frac{9}{4} \left(\int_{\mathbb{B}^4} |\nabla_{\mathbb{H}} u|^2 dV - \frac{9}{4} \int_{\mathbb{B}^4} |u|^2 dV \right). \end{aligned}$$

Combing (5.1) and the following Hardy-Sobolev inequality on \mathbb{B}^4 (see e.g. [27])

$$(5.2) \quad \int_{\mathbb{B}^4} |\nabla_{\mathbb{H}} u|^2 dV - \frac{9}{4} \int_{\mathbb{B}^4} |u|^2 dV \geq C_4 \left(\int_{\mathbb{B}^4} |u|^4 dV \right)^{\frac{1}{2}},$$

we have,

$$(5.3) \quad 1 \geq \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + \alpha) u \cdot u dV \geq C_5^{-1} \left(\int_{\mathbb{B}^4} |u|^4 dV \right)^{\frac{1}{2}},$$

where $C_5 = \frac{4}{9C_4}$.

Set $\Omega(u) = \{x \in \mathbb{B}^4 : |u(x)| \geq 1\}$. By (5.3), we have

$$(5.4) \quad |\Omega(u)| = \int_{\Omega(u)} dV \leq \int_{\mathbb{B}} |u(z)|^4 dV \leq C_5^2.$$

We write

$$(5.5) \quad \begin{aligned} & \int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV \\ &= \int_{\Omega(u)} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV + \int_{\mathbb{B}^4 \setminus \Omega(u)} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV \\ &\leq \int_{\Omega(u)} e^{32\pi^2 u^2} dV + \int_{\mathbb{B}^4 \setminus \Omega(u)} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV. \end{aligned}$$

Notice that on the domain $\mathbb{B}^4 \setminus \Omega(u)$, we have $|u(x)| < 1$. Thus,

$$(5.6) \quad \begin{aligned} & \int_{\mathbb{B}^4 \setminus \Omega(u)} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV \\ &= \int_{\mathbb{B}^4 \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(32\pi^2 u^2)^n}{n!} dV \\ &\leq \int_{\mathbb{B}^4 \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(32\pi^2)^n u^4}{n!} dV \\ &\leq \sum_{n=2}^{\infty} \frac{(32\pi^2)^n}{n!} \int_{\mathbb{B}^4} |u(x)|^4 dV \\ &\leq e^{32\pi^2} C_5^2. \end{aligned}$$

To finish the proof, it is enough to show $\int_{\Omega(u)} e^{32\pi^2 u^2} dV$ is bounded by some universal constant. By (5.4), we may assume

$$(5.7) \quad |\Omega(u)| \leq \Omega_0$$

for some constant Ω_0 which is independent of u . Now rewrite

$$(5.8) \quad \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + \alpha) u \cdot u dV = \int_{\mathbb{B}^4} \left| \sqrt{(-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + \alpha)} u \right|^2 dV$$

and set

$$v = \sqrt{(-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + \alpha)} u = \sqrt{-\Delta_{\mathbb{H}} - 9/4} (\sqrt{-\Delta_{\mathbb{H}} + \alpha} u).$$

Then

$$(5.9) \quad \int_{\mathbb{B}^4} |v|^2 dV = \int_{\mathbb{B}^4} \left| \sqrt{-\Delta_{\mathbb{H}} - 9/4} (\sqrt{-\Delta_{\mathbb{H}} + \alpha} u) \right|^2 dV \leq 1.$$

Furthermore, by (2.2), we can write u as a potential

$$u = (v * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}) * (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} = v * ((-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}}).$$

Let $\varphi_1 = (-\Delta_{\mathbb{H}} - 9/4)^{-\frac{1}{2}} * (-\Delta_{\mathbb{H}} + \alpha)^{-\frac{1}{2}}$. Then $u = v * \varphi_1$. By (5.7) and O'Neil's lemma,

$$\begin{aligned} \int_{\Omega(u)} e^{32\pi^2 u^2} dV &= \int_0^{|\Omega(u)|} \exp(32\pi^2 |u^*(t)|^2) dt \leq \int_0^{\Omega_0} \exp(32\pi^2 |u^*(t)|^2) dt \\ &\leq \int_0^{\Omega_0} \exp \left(32\pi^2 \left| \frac{1}{t} \int_0^t v^*(s) ds \int_0^t \varphi_1^*(s) ds + \int_t^\infty v^*(s) \varphi_1^*(s) ds \right|^2 \right) dt \\ &= \Omega_0 \int_0^\infty \exp \left(-t + 32\pi^2 \left| \frac{1}{\Omega_0 e^{-t}} \int_0^{\Omega_0 e^{-t}} v^*(s) ds \int_0^{\Omega_0 e^{-t}} \varphi_1^*(s) ds + \int_{\Omega_0 e^{-t}}^\infty v^*(s) \varphi_1^*(s) ds \right|^2 \right) dt. \end{aligned}$$

To get the last equation, we use the substitution $t := \Omega_0 e^{-t}$. Next, we change the variables

$$\begin{aligned} \psi(t) &= \sqrt{\Omega_0 e^{-t}} v^*(\Omega_0 e^{-t}); \\ \varphi(t) &= \sqrt{32\pi^2 \Omega_0 e^{-t}} \varphi_1^*(\Omega_0 e^{-t}). \end{aligned}$$

It is easy to check

$$\begin{aligned} \int_t^\infty e^{-s/2} \psi(s) ds \int_t^\infty e^{-s/2} \varphi(s) ds &= \frac{\sqrt{32\pi^2}}{\Omega_0} \int_0^{\Omega_0 e^{-t}} v^*(s) ds \int_0^{\Omega_0 e^{-t}} \varphi_1^*(s) ds; \\ \int_{-\infty}^t \psi(s) \varphi(s) ds &= \sqrt{32\pi^2} \int_{\Omega_0 e^{-t}}^\infty v^*(s) \varphi_1^*(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Omega(u)} e^{32\pi^2 u^2} dV &\leq \int_0^{\Omega_0} \exp(32\pi^2 |u^*(t)|^2) dt \\ &= \Omega_0 \int_0^\infty \exp \left(-t + \left(e^t \int_t^\infty e^{-s/2} \psi(s) ds \int_t^\infty e^{-s/2} \varphi(s) ds + \int_{-\infty}^t \psi(s) \varphi(s) ds \right)^2 \right) dt \\ &= \Omega_0 \int_0^\infty e^{-F(t)} dt, \end{aligned}$$

where

$$F(t) = t - \left(e^t \int_t^\infty e^{-s/2} \psi(s) ds \int_t^\infty e^{-s/2} \varphi(s) ds + \int_{-\infty}^t \psi(s) \varphi(s) ds \right)^2.$$

Set

$$(5.10) \quad a(s, t) = \begin{cases} \varphi(s), & s < t; \\ e^t \left(\int_t^\infty e^{-r/2} \varphi(r) dr \right) e^{-s/2}, & s > t. \end{cases}$$

Then

$$F(t) = t - \left(\int_{-\infty}^\infty a(s, t) \psi(s) ds \right)^2.$$

To complete the proof, we need to show that there exists a constant C which is independent of ψ such that

$$\int_0^\infty e^{-F(t)} dt < C.$$

This will be done in the following Lemma 5.1. The proof of Theorem 1.6 is thereby completed.

Lemma 5.1. *Let $\psi(s)$, $a(s, t)$ and $F(t)$ be as above. Then there a constant C_6 which is independent of ψ such that $\int_0^\infty e^{-F(t)} dt < C_6$.*

Proof. The proof is similar to that in Adams' paper [1]. Notice that

$$\int_0^\infty e^{-F(t)} dt = \int_{-\infty}^\infty |E_\lambda| e^{-\lambda} d\lambda,$$

where $E_\lambda = \{t \geq 0 : F(t) \leq \lambda\}$ and $|E_\lambda|$ is the Lebesgue measure of E_λ . It is enough to show the following two facts:

- (a) There exists a constant $c \geq 0$ which is independent of ψ such that $\inf_{t \geq 0} F(t) \geq -c$;
- (b) There exist constants B_1 and B_2 which are both independent of ψ such that $|E_\lambda| \leq B_1|\lambda| + B_2$.

Firstly, we prove (a). Without loss of generality, we assume $\Omega_0 > 2$ so that $\ln \frac{\Omega_0}{2} > 0$. By Lemma 4.3,

$$(5.11) \quad \begin{aligned} \varphi(t) &= \sqrt{32\pi^2\Omega_0 e^{-t}} \varphi_1^*(\Omega_0 e^{-t}) \leq 1 + A_1 \sqrt[4]{\Omega_0} e^{-\frac{t}{4}}, \quad t \in \mathbb{R}; \\ \varphi(t) &= \sqrt{32\pi^2\Omega_0 e^{-t}} \varphi_1^*(\Omega_0 e^{-t}) \lesssim \frac{1}{\ln \Omega_0 - t} \leq \frac{C}{1-t}, \quad t \leq \ln \frac{\Omega_0}{2}, \end{aligned}$$

where $C > 0$ is a constant which is independent of ψ . Therefore, by (5.10), for $t \geq 0$,

$$\begin{aligned} \int_{-\infty}^\infty a(s, t)^2 ds &= \int_{-\infty}^0 a(s, t)^2 ds + \int_0^t a(s, t)^2 ds + \int_t^\infty a(s, t)^2 ds \\ &= \int_{-\infty}^0 \varphi^2(s) ds + \int_0^t \varphi^2(s) ds + e^{2t} \left(\int_t^\infty e^{-r/2} \varphi(r) dr \right)^2 \int_t^\infty e^{-s} ds \\ &\leq C \int_{-\infty}^0 \frac{1}{(1-s)^2} ds + \int_0^t (1 + A_1 \sqrt[4]{\Omega_0} e^{-\frac{s}{4}})^2 ds + \\ &\quad e^t \left(\int_t^\infty e^{-r/2} (1 + A_1 \sqrt[4]{\Omega_0} e^{-\frac{r}{4}}) dr \right)^2 \\ &\leq t + c, \end{aligned}$$

where $c > 0$ is independent of ψ . Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned}
F(t) &= t - \left(\int_{-\infty}^{\infty} a(s, t) \psi(s) ds \right)^2 \\
&\geq t - \int_{-\infty}^{\infty} a(s, t)^2 ds \int_{-\infty}^{\infty} \psi^2(s) ds \\
&= t - \int_{-\infty}^{\infty} a(s, t)^2 ds \int_{-\infty}^{\infty} \Omega_0 e^{-t} |v^*(\Omega_0 e^{-t})|^2 ds \\
&\geq t - (t + c) \int_{\mathbb{B}^4} |v|^2 dV \\
&= t - (t + c) = -c.
\end{aligned}$$

Secondly, we prove (b). Let $R > 0$ and suppose $E_\lambda \cap [R, \infty) \neq \emptyset$. Take $t_1, t_2 \in E_\lambda \cap [R, \infty) \neq \emptyset$, $t_1 < t_2$. Then

$$\begin{aligned}
(5.12) \quad t_2 - \lambda &\leq \left(\int_{-\infty}^{\infty} a(s, t_2) \psi(s) dt \right)^2 \\
&= \left(\int_{-\infty}^{t_1} a(s, t_2) \psi(s) ds + \int_{t_1}^{t_2} a(s, t_2) \psi(s) ds + \int_{t_2}^{\infty} a(s, t_2) \psi(s) ds \right)^2 \\
&= \left(\int_{-\infty}^{t_1} \varphi(s) \psi(s) ds + \int_{t_1}^{t_2} \varphi(s) \psi(s) ds + \int_{t_2}^{\infty} a(s, t_2) \psi(s) ds \right)^2.
\end{aligned}$$

Set $L(t) = \left(\int_t^{\infty} \psi^2(s) ds \right)^{\frac{1}{2}}$. Then

$$L(t) \leq \left(\int_{-\infty}^{\infty} \psi^2(s) ds \right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} \Omega_0 e^{-t} |v^*(\Omega_0 e^{-t})|^2 ds \right)^{\frac{1}{2}} = \left(\int_{\mathbb{B}^4} |v|^2 dV \right)^{\frac{1}{2}} \leq 1$$

Therefore, by (5.11) and Cauchy-Schwarz inequality,

$$\begin{aligned}
(5.13) \quad \left(\int_{-\infty}^{t_1} \varphi(s) \psi(s) ds \right)^2 &\leq \int_{-\infty}^{t_1} \varphi^2(s) ds \cdot \int_{-\infty}^{\infty} \psi^2(s) ds \\
&\leq \int_{-\infty}^{t_1} \varphi^2(s) ds = \int_{-\infty}^0 \varphi^2(s) ds + \int_0^{t_1} \varphi^2(s) ds \\
&\leq C \int_{-\infty}^0 \frac{1}{(1-s)^2} ds + \int_0^{t_1} (1 + A_1 \sqrt[4]{\Omega_0} e^{-\frac{s}{4}})^2 ds \\
&\leq t_1 + b_1;
\end{aligned}$$

$$\begin{aligned}
(5.14) \quad & \left(\int_{t_1}^{t_2} \varphi(s) \psi(s) ds \right)^2 \leq \int_{t_1}^{t_2} \varphi^2(s) ds \cdot L^2(t_1) \\
& \leq \int_{t_1}^{t_2} (1 + A_1^4 \sqrt{\Omega_0} e^{-\frac{s}{4}})^2 ds \cdot L^2(t_1) \\
& \leq (t_2 - t_1 + b_2) L^2(t_1); \\
(5.15) \quad & \left(\int_{t_2}^{\infty} a(s, t_2) \psi(s) ds \right)^2 \leq \int_{t_2}^{\infty} a(s, t_2)^2 ds \cdot L^2(t_2) \leq \int_{t_2}^{\infty} a(s, t_2)^2 ds \cdot L^2(t_1) \\
& = e^{2t_2} \left(\int_{t_2}^{\infty} e^{-r/2} \varphi(r) dr \right)^2 \int_{t_2}^{\infty} e^{-s} ds \cdot L^2(t_1) \\
& \leq e^{t_2} \left(\int_{t_2}^{\infty} e^{-r/2} ((1 + A_1^4 \sqrt{\Omega_0} e^{-\frac{r}{4}}) dr \right)^2 \cdot L^2(t_1) \\
& \leq b_3 L^2(t_1),
\end{aligned}$$

where b_1, b_2, b_3 are constants independent of t_1 and t_2 . Combing (5.12) and (5.13)-(5.15) yields

$$\begin{aligned}
t_2 - \lambda & \leq \left\{ (t_1 + b_1)^{\frac{1}{2}} + [(t_2 - t_1 + b_2)^{\frac{1}{2}} + b_3] L(t_1) \right\}^2 \\
& \leq \left\{ (t_1 + b_1)^{\frac{1}{2}} + [(t_2 - t_1)^{\frac{1}{2}} + b_4] L(t_1) \right\}^2,
\end{aligned}$$

where b_4 is a constant independent of t_1 and t_2 . The rest of the proof is the same as that in [1] and thus the proof is completed. \square

Proof of Theorem 1.8 Let $u \in C_0^\infty(\mathbb{B}^4)$ be such that

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx - \lambda \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \leq 1.$$

Then

$$(9 - \lambda) \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \leq \int_{\mathbb{B}^4} |\Delta u|^2 dx - \lambda \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \leq 1.$$

Therefore, by Corollary 1.7,

$$\begin{aligned}
\int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1) dV & = 16 \int_{\mathbb{B}^4} \frac{e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2}{(1 - |x|^2)^4} dx + 16 \cdot 32\pi^2 \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \\
& \leq C_1 + \frac{16 \cdot 32\pi^2}{9 - \lambda}.
\end{aligned}$$

The desired results follows.

Before the proof of Theorem 1.9, we need the following improved Hardy inequality.

Lemma 5.2. *There exists a constant $C_7 > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^4)$,*

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx - 9 \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \geq C_7 \int_{\mathbb{B}^4} u^2 dx.$$

Proof. It is enough to show

$$\int_{\mathbb{B}^4} |\nabla_{\mathbb{H}} u|^2 dV - \frac{9}{4} \int_{\mathbb{B}^4} u^2 dV \geq C_8 \int_{\mathbb{B}^4} u^2 dx,$$

since, by Plancherel formula,

$$\begin{aligned} & \int_{\mathbb{B}^4} |\Delta u|^2 dx - 9 \int_{\mathbb{B}^4} \frac{u^2}{(1 - |x|^2)^4} dx \\ &= \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4) (-\Delta_{\mathbb{H}} + 1/2) u \cdot u dV \\ &= D_4 \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{n-1}} \frac{\lambda^2}{4} \cdot \left(\frac{9 + \lambda^2}{4} + \frac{1}{2} \right) |\widehat{f}(\lambda, \zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\zeta) \\ &\geq \left(\frac{9}{4} + \frac{1}{2} \right) D_4 \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{n-1}} \frac{\lambda^2}{4} |\widehat{f}(\lambda, \zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\zeta) \\ &= \left(\frac{9}{4} + \frac{1}{2} \right) \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 9/4) u \cdot u dV \\ &= \left(\frac{9}{4} + \frac{1}{2} \right) \left(\int_{\mathbb{B}^4} |\nabla_{\mathbb{H}} u|^2 dV - \frac{9}{4} \int_{\mathbb{B}^4} |u|^2 dV \right). \end{aligned}$$

Set $u = (1 - |x|^2)f$. Then $f \in C_0^\infty(\mathbb{B}^4)$ and

$$|\nabla u|^2 = (1 - |x|^2)^2 |\nabla f|^2 + \frac{1}{2} \langle \nabla f^2, \nabla(1 - |x|^2)^2 \rangle + f^2 \cdot 4|x|^2.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{B}^4} \frac{|\nabla u|^2}{(1 - |x|^2)^2} dx &= \int_{\mathbb{B}^4} |\nabla f|^2 dx + \int_{\mathbb{B}^4} f^2 \frac{4|x|^2}{(1 - |x|^2)^2} dx - \frac{1}{2} \int_{\mathbb{B}^4} f^2 \Delta \ln(1 - |x|^2)^2 dx \\ &= \int_{\mathbb{B}^4} |\nabla f|^2 dx + 8 \int_{\mathbb{B}^4} \frac{f^2}{(1 - |x|^2)^2} dx. \end{aligned}$$

Recall that the improved Hardy inequality (see e.g.[6, 37])

$$\int_{\mathbb{B}^4} |\nabla f|^2 dx - \int_{\mathbb{B}^4} \frac{f^2}{(1 - |x|^2)^2} dx \geq C_9 \int_{\mathbb{B}^4} f^2 dx.$$

We have

$$\begin{aligned}
\int_{\mathbb{B}^4} |\nabla_{\mathbb{H}} u|^2 dV - \frac{9}{4} \int_{\mathbb{B}^4} u^2 dV &= 4 \left(\int_{\mathbb{B}^4} \frac{|\nabla u|^2}{(1-|x|^2)^2} dx - 9 \int_{\mathbb{B}^4} \frac{u^2}{(1-|x|^2)^2} dx \right) \\
&= 4 \left(\int_{\mathbb{B}^4} |\nabla f|^2 dx - \int_{\mathbb{B}^4} \frac{f^2}{(1-|x|^2)^2} dx \right) \\
&\geq 4C_9 \int_{\mathbb{B}^4} f^2 dx = 4C_9 \int_{\mathbb{B}^4} \frac{u^2}{1-|x|^2} dx \\
&\geq 4C_9 \int_{\mathbb{B}^4} u^2 dx.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.9 Let $u \in C_0^\infty(\mathbb{B}^4)$ be such that

$$\int_{\mathbb{B}^4} |\Delta u|^2 dx - 9 \int_{\mathbb{B}^4} \frac{u^2}{(1-|x|^2)^4} dx \leq 1,$$

By Corollary 1.5, there exist a positive constant $C_1 > 0$ which is independent of u such that

$$\int_{\mathbb{B}^4} \frac{(e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2)}{(1-|x|^2)^4} dx \leq C_1.$$

Therefore, by Lemma 5.2,

$$\begin{aligned}
\int_{\mathbb{B}^4} e^{32\pi^2 u^2} dx &= \int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dx + \int_{\mathbb{B}^4} dx + 32\pi^2 \int_{\mathbb{B}^4} u^2 dx \\
&\leq \int_{\mathbb{B}^4} \frac{(e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2)}{(1-|x|^2)^4} dx + \int_{\mathbb{B}^4} dx + 32\pi^2 \int_{\mathbb{B}^4} u^2 dx \\
&\leq C_1 + \int_{\mathbb{B}^4} dx + 32\pi^2 C_7^{-1}.
\end{aligned}$$

The desired result follows.

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